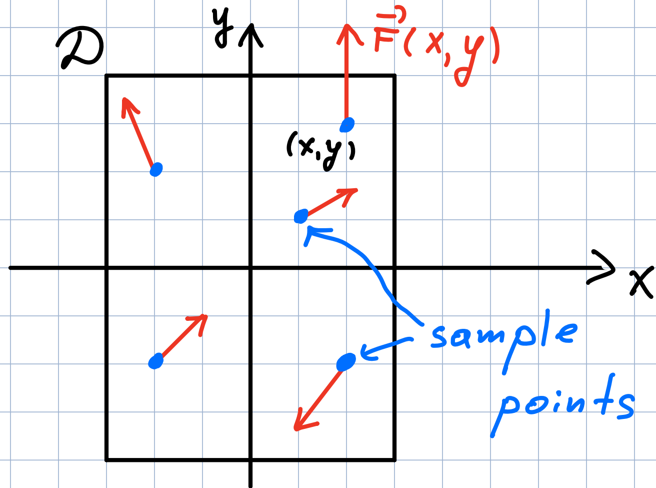
LAST TIME:

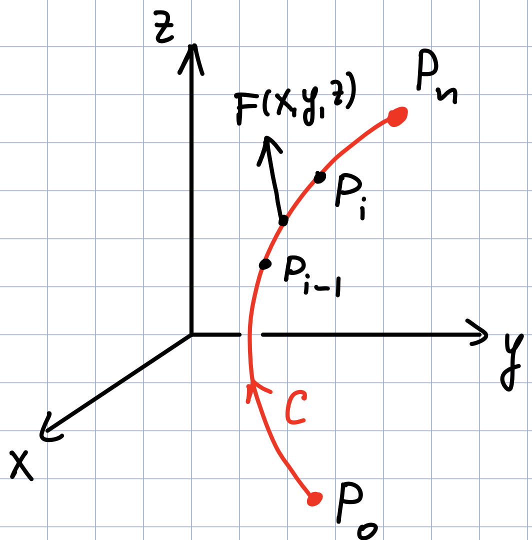


A vector field on $\mathbb{R}^2$ is a function

$$\vec{F}(x,y)$$

assigning to each point $(x,y) \in \mathcal{D}$ a 2D vector $\vec{F}(x,y)$

region in $\mathbb{R}^2$



C - curve , $\vec{F}(x,y,z)$ - force field

Work W done by force $\vec{F}$ while moving a particle along curve C:

$$\boxed{W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

Line integral of a vector field $\vec{F}$ along a curve C given by $\vec{r}(t)$, $a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\vec{F}(x(t),\, y(t),\, z(t))} \cdot \vec{r}\,'(t)\,dt$$

Fundamental theorem of line integrals

Recall: a) Fund. thm. of Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

b) gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$

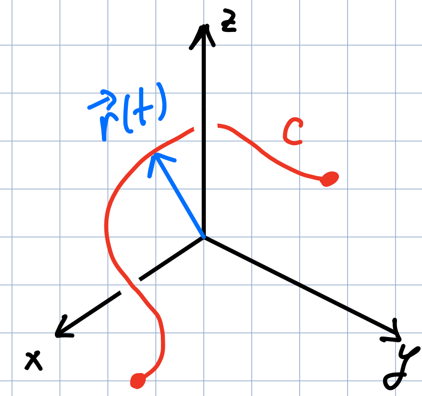
"potential function" of $\vec{F}$

Rmk: A vector field with a potential is called "conservative".

Fundamental THM of line integrals:

Let C be a curve (on a plane or in space) given by $\vec{r}(t)$, $a \le t \le b$. Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$



Ex: Gravitational field is $\vec{F}(\vec{r}) = -\frac{k\vec{r}}{|\vec{r}|^3}$, where k is a const.
Find the work done by grav. field when moving a particle from $(3,4,12)$ to $(2,2,0)$ along some curve C.

Sol: 1) Note that $-\frac{k\vec{r}}{|\vec{r}|^3} = \nabla \frac{k}{|\vec{r}|}$

// indeed,

$$f(x,y) = \frac{k}{\sqrt{x^2+y^2}} \Rightarrow f_x(x,y) = -\frac{1}{2}\frac{k\cdot 2x}{(x^2+y^2)^{3/2}}$$

$$f_y(x,y) = -\frac{1}{2}\frac{k\cdot 2y}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \nabla f(x,y) = \left\langle -\frac{kx}{(x^2+y^2)^{3/2}}, -\frac{ky}{(x^2+y^2)^{3/2}} \right\rangle = \frac{-k}{(x^2+y^2)^{3/2}}\langle x,y\rangle = \frac{-k}{|\vec{r}|^3}\vec{r} \quad //$$

2) $$W = \int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,2,0) - f(3,4,12)$$

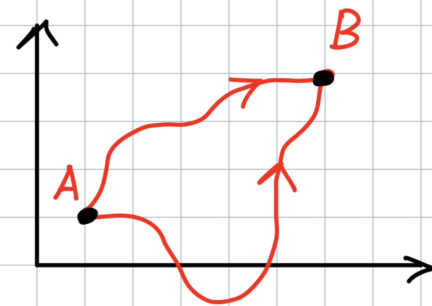
$$= \frac{k}{\sqrt{8}} - \frac{k}{\sqrt{\underbrace{3^2+4^2+12^2}_{169}}} = k\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

$\vec{F}$ - vector field. The following are equivalent:

(1) $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$

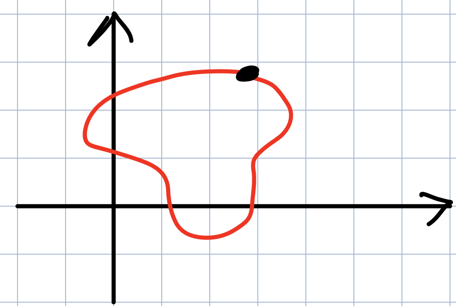
(2) $\int_C \vec{F} \cdot d\vec{r}$ is the same for all curves with initial point A and final point B

("independent of path")



(3) $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve

(starting and ending at the same point)



Let $\vec{F}(x,y)$ be a vector field in a simply-connected region D

single-piece, without holes

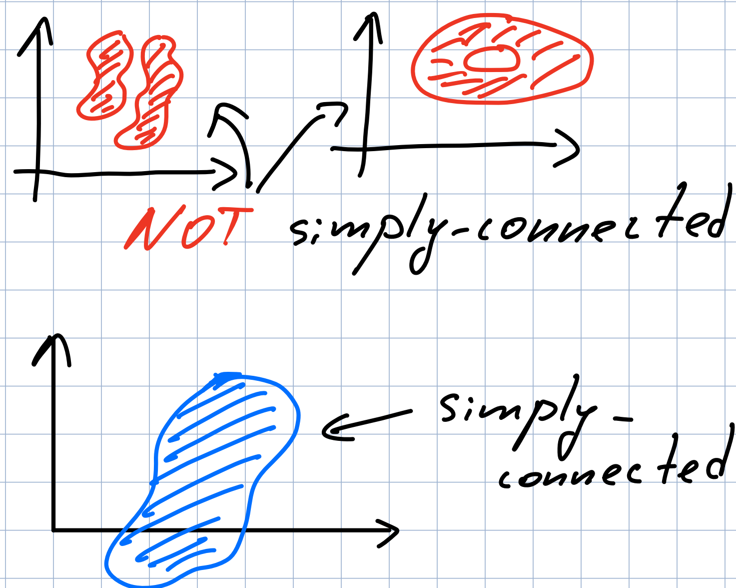
$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

THEN $\vec{F}(x,y)$ is conservative $\underbrace{\text{iff}}_{\text{if and only if}}$ $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$

Idea: 1) If $F = \nabla f$ then $P = f_x$, $Q = f_y$ $\Rightarrow$ $\underbrace{P_y}_{= \frac{\partial P}{\partial y}} = f_{xy} = f_{yx} = \underbrace{Q_x}_{= \frac{\partial Q}{\partial x}}$

2) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \Rightarrow$ conservative

consequence of Greens thm (next week)



Ex: $\vec{F}(x,y) = \langle \underbrace{x-y}_{P}, \underbrace{x-z}_{Q} \rangle$. Is it conservative?

Sol: $P_y = -1$, $Q_x = 1 \Rightarrow P_y \neq Q_x \rightsquigarrow$ NO.

Ex: $\vec{F}(x,y) = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2-3y^2}_{Q} \rangle$. Is it conservative?

Sol: $P_y = 2x$
$Q_x = 2x$ $\Rightarrow$ YES

In space: $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ is conservative iff

$$\begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$$

Ex: $\vec{F}(x,y) = \langle 3+2xy, x^2-3y^2 \rangle$

a) Find f such that $\vec{F} = \nabla f$

b) Find $\int_C \vec{F}\cdot d\vec{r}$, where C is a curve given by $\vec{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \le t \le \pi$.

Sol: (a) WANT f such that

$f_x = 3+2xy$ (integrate w.r.t. x) $f(x,y) = 3x + x^2y + g(y)$

$f_y = x^2 - 3y^2$

$\Rightarrow f_y(x,y) = 0 + x^2 + g'(y) = x^2 - 3y^2 \Rightarrow g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + \underbrace{K}_{\text{constant}}$

thus $\boxed{f(x,y) = 3x + x^2y - y^3 + K}$

(b) Curve C starts at $\vec{r}(0) = \langle 0, 1 \rangle$ and ends at $\vec{r}(\pi) = \langle 0, -e^{\pi} \rangle$

Therefore, $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \underbrace{f(0,-e^{\pi})}_{f(\vec{r}(\pi))} - \underbrace{f(0,1)}_{f(\vec{r}(0))} = (e^{3\pi}+K) - (-1+K) = e^{3\pi}+1$

Ex: $\vec{F}(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$. Find f such that $\vec{F} = \nabla f$

Sol:

$f_x = y^2 \overset{\int}{\Longrightarrow} f(x,y,z) = xy^2 + g(y,z)$

$f_y = 2xy + e^{3z}$ $\quad f_y = 2xy + g_y(y,z) = 2xy + e^{3z} \Rightarrow g_y = e^{3z}$

$f_z = 3ye^{3z}$ $\quad \overset{\int}{\Longrightarrow} g = ye^{3z} + h(z)$

$f_z = 0 + g_z(y,z) = 3ye^{3z} + h_z = 3ye^{3z} \Rightarrow h_z = 0 \Rightarrow h = \underbrace{K}_{\text{const}}$

$$\Rightarrow \boxed{f(x,y,z) = xy^2 + ye^{3z} + K}$$